4.6 Quadrature Amplitude Modulation (QAM)

4.68. We are now going to define a quantity called the "bandwidth" of a signal. Unfortunately, in practice, there isn't just one definition of bandwidth.

Definition 4.69. The **bandwidth** (BW) of a signal is usually calculated from the differences between two frequencies (called the bandwidth limits). Let's consider the following definitions of bandwidth for real-valued signals [3, p 173]

$f_{max} - f_{min}$ f_{max} f_{min}

(a) **Absolute bandwidth**: Use the highest frequency and the lowest frequency in the positive-f part of the signal's nonzero magnitude spectrum.

- This uses the frequency range where 100% of the energy is confined.
- We can speak of absolute bandwidth if we have ideal filters and unlimited time signals.

(b) **3-dB bandwidth** (**half-power bandwidth**): Use the frequencies where the signal power starts to decrease by 3 dB (1/2).

- The magnitude is reduced by a factor of $1/\sqrt{2}$.

(c) **Null-to-null bandwidth**: Use the signal spectrum's first set of zero crossings.

(d) **Occupied bandwidth**: Consider the frequency range in which X% (for example, 99%) of the energy is contained in the signal's bandwidth.

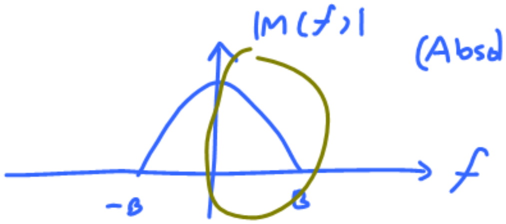
(e) **Relative power spectrum bandwidth**: the level of power outside the bandwidth limits is reduced to some value relative to its maximum level.

- Usually specified in negative decibels (dB).
- For example, consider a 200-kHz-BW broadcast signal with a maximum carrier power of 1000 watts and relative power spectrum bandwidth of -40 dB (i.e., 1/10,000). We would expect the station's power emission to not exceed 0.1 W outside of $f_c \pm 100$ kHz.

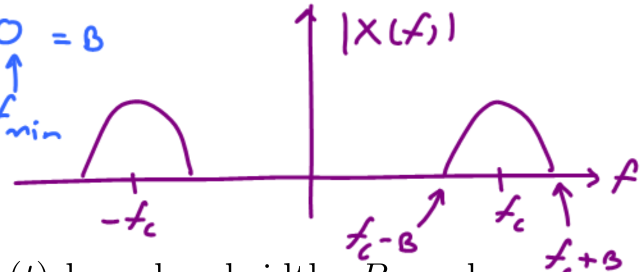
(Absolute) BW $= f_c + B - (f_c - B) = 2B$

$x(t) = m(t)\cos(2\pi f_c t)$

Example 4.70. Message bandwidth and the transmitted signal bandwidth



(Absolute) BW $= \underbrace{B}_{f_{max}} - \underbrace{0}_{f_{min}} = B$



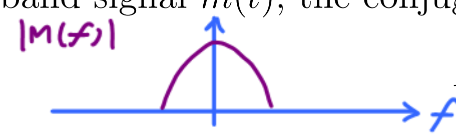
4.71. Rough Approximation: If $g_1(t)$ and $g_2(t)$ have bandwidths B_1 and B_2 Hz, respectively, the bandwidth of $g_1(t)g_2(t)$ is $B_1 + B_2$ Hz.

This result follows from the application of the width property[18] of convolution[19] to the convolution-in-frequency property.

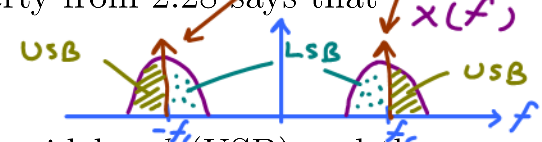
Consequently, if the bandwidth of $g(t)$ is B Hz, then the bandwidth of $g^2(t)$ is $2B$ Hz, and the bandwidth of $g^n(t)$ is nB Hz. We mentioned this property in 2.38.

If using AM, then also have these carrier components

4.72. BW Inefficiency in DSB-SC system: Recall that for real-valued baseband signal $m(t)$, the conjugate symmetry property from 2.28 says that



$$M(-f) = (M(f))^* .$$

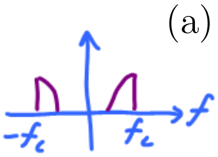


The DSB spectrum has two sidebands: the upper sideband (USB) and the lower sideband (LSB), both containing complete information about the baseband signal $m(t)$. As a result, DSB signals occupy twice the bandwidth required for the baseband.

Absolute BW = 2B

4.73. To improve the spectral efficiency of amplitude modulation, there exist two basic schemes to either utilize or remove the spectral redundancy:

difficult to do directly



(a) Single-sideband (SSB) modulation, which removes either the LSB or the USB so that for one message signal $m(t)$, there is only a bandwidth of B Hz. There

Absolute BW = B

(b) Quadrature amplitude modulation (QAM), which utilizes spectral redundancy by sending two messages over the same bandwidth of $2B$ Hz.

We will only discussed QAM here. SSB discussion can be found in [3, Sec 4.4], [13, Section 3.1.3] and [4, Section 4.5].

[18]This property states that the width of $x * y$ is the sum of the widths of x and y.

[19]The width property of convolution does not hold in some pathological cases. See [4, p 98].

Definition 4.74. In ***quadrature amplitude modulation*** (***QAM***) or ***quadrature multiplexing***, two baseband real-valued signals $m_1(t)$ and $m_2(t)$ are transmitted simultaneously via the corresponding QAM signal:

$$x_{\mathrm{QAM}}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t). \quad \leftarrow \text{Form \#1}$$

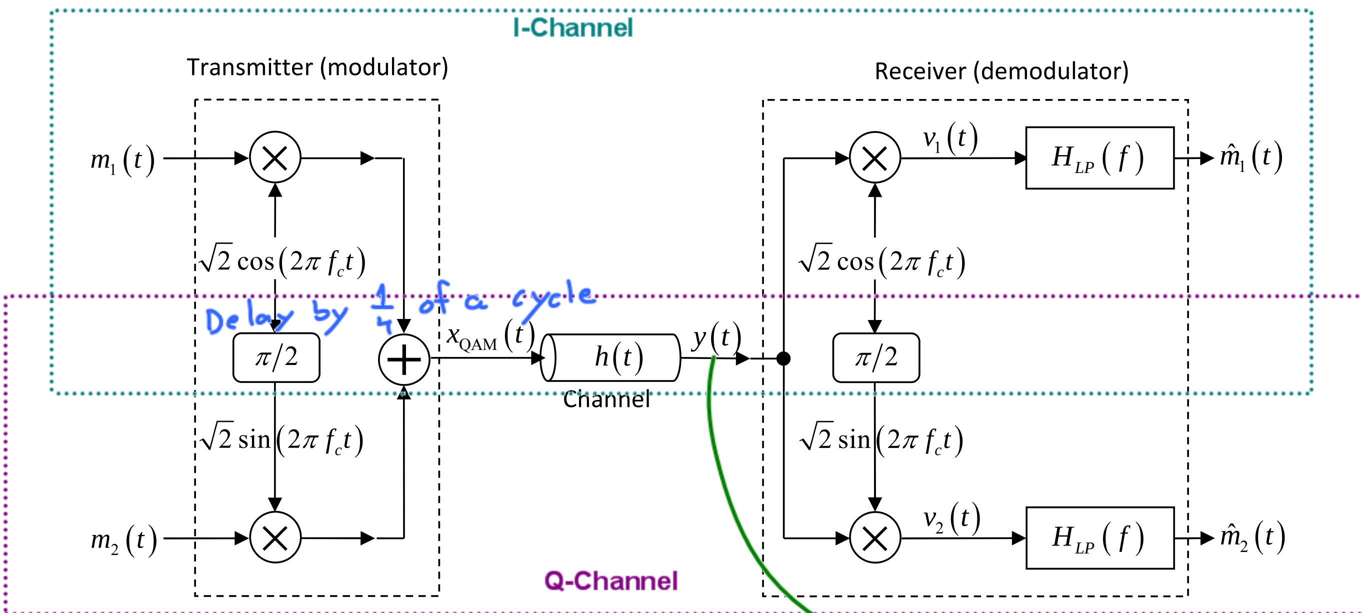


Figure 23: QAM Scheme

when $h(t) = \delta(t)$,
$y(t) = x_{QAM}(t)$.

- QAM operates by transmitting two DSB signals via carriers of the same frequency but in phase quadrature.
- Both modulated signals simultaneously occupy the same frequency band.
- The "cos" (upper) channel is also known as the ***in-phase*** (***I***) channel and the "sin" (lower) channel is the ***quadrature*** (***Q***) channel.

4.75. ***Demodulation***: The two baseband signals can be separated at the receiver by synchronous detection: assuming $h(t) = \delta(t)$

$$\hat{m}_1(t) = \mathrm{LPF}\left\{\overbrace{\underbrace{x_{\mathrm{QAM}}(t)}_{y(t)}\sqrt{2}\cos(2\pi f_c t)}^{v_1(t)}\right\} = m_1(t)$$
$$\mathrm{LPF}\left\{x_{\mathrm{QAM}}(t)\sqrt{2}\sin(2\pi f_c t)\right\} = m_2(t)$$

$v_1(t) = \left(m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)\right)\sqrt{2}\cos(2\pi f_c t)$

$= m_1(t)\, 2\cos^2(2\pi f_c t) + m_2(t)\, 2\sin(2\pi f_c t)\cos(2\pi f_c t)$

$= m_1(t)\left(1 + \cos(2\pi(2f_c)t)\right) + m_2(t)\sin\left(2\pi(2f_c)t\right)$

$= m_1(t) + \underbrace{m_1(t)\cos(2\pi(2f_c)t)}_{\to 0\ (\mathrm{LPF})} + \underbrace{m_2(t)\cos(2\pi(2f_c)t - 90^\circ)}_{\to 0\ (\mathrm{LPF})}$

Derivation of $2\cos\alpha\sin\alpha = \sin(2\alpha)$

Use
$$\left.\begin{aligned}\cos\alpha &= \frac{e^{j\alpha}+e^{-j\alpha}}{2}\\ \sin\alpha &= \frac{e^{j\alpha}-e^{-j\alpha}}{2j}\end{aligned}\right\} \Rightarrow 2\cos\alpha\sin\alpha = \cancel{2}\underbrace{\left(\frac{e^{j\alpha}+e^{-j\alpha}}{\cancel{2}}\right)}_{(A+B)}\underbrace{\left(\frac{e^{j\alpha}-e^{-j\alpha}}{2j}\right)}_{(A-B)}$$

$$= \frac{A^2-B^2}{2j} = \frac{e^{j(2\alpha)}-e^{-j(2\alpha)}}{2j} = \sin(2\alpha)$$

Alternative form for $x_{QAM}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)$

At a particular time t (or over time interval $t_1 \le t \le t_2$ where $m_1(t)$ and $m_2(t)$ are constant),

suppose $m_1(t) \equiv m_1$ and $m_2(t) \equiv m_2$.

Then $x_{QAM}(t) = m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\sin(2\pi f_c t)$

$$= m_1\sqrt{2}\cos(2\pi f_c t) + m_2\sqrt{2}\cos(2\pi f_c t - 90^\circ)$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

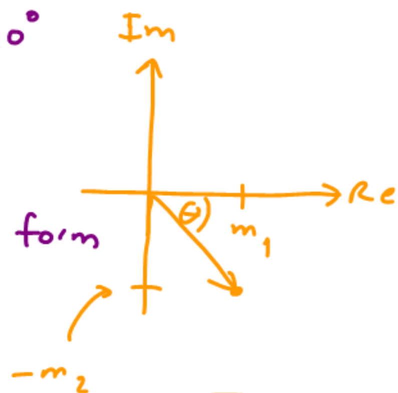
using the phasor representation to combine the two sinusiods

$$m_1\sqrt{2}\angle 0^\circ + m_2\sqrt{2}\angle -90^\circ$$

$$\parallel$$

$$\sqrt{2}(m_1 - jm_2)$$

$$= E\angle\theta \leftarrow \text{polar form}$$



$$\Updownarrow$$

$$= E\cos(2\pi f_c t + \theta)$$

$\tan^{-1}\left(\frac{-m_2}{m_1}\right)$

$-\tan^{-1}\left(\frac{m_2}{m_1}\right)$

When m_1 and m_2 can only be $+1$ or -1,

we can make a table for this:

m_1	m_2	$m_1 - jm_2$	$\sqrt{2}(m_1 - jm_2)$
1	1	$1-j = \sqrt{2}\angle -45^\circ$	$2\angle -45^\circ$
1	-1	$1+j = \sqrt{2}\angle 45^\circ$	$2\angle 45^\circ$
-1	1	$-1-j = \sqrt{2}\angle -135^\circ$	$2\angle -135^\circ$
-1	-1	$-1+j = \sqrt{2}\angle 135^\circ$	$2\angle 135^\circ$

Quiz 2 Solution

Suppose $x_{QAM}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)$.

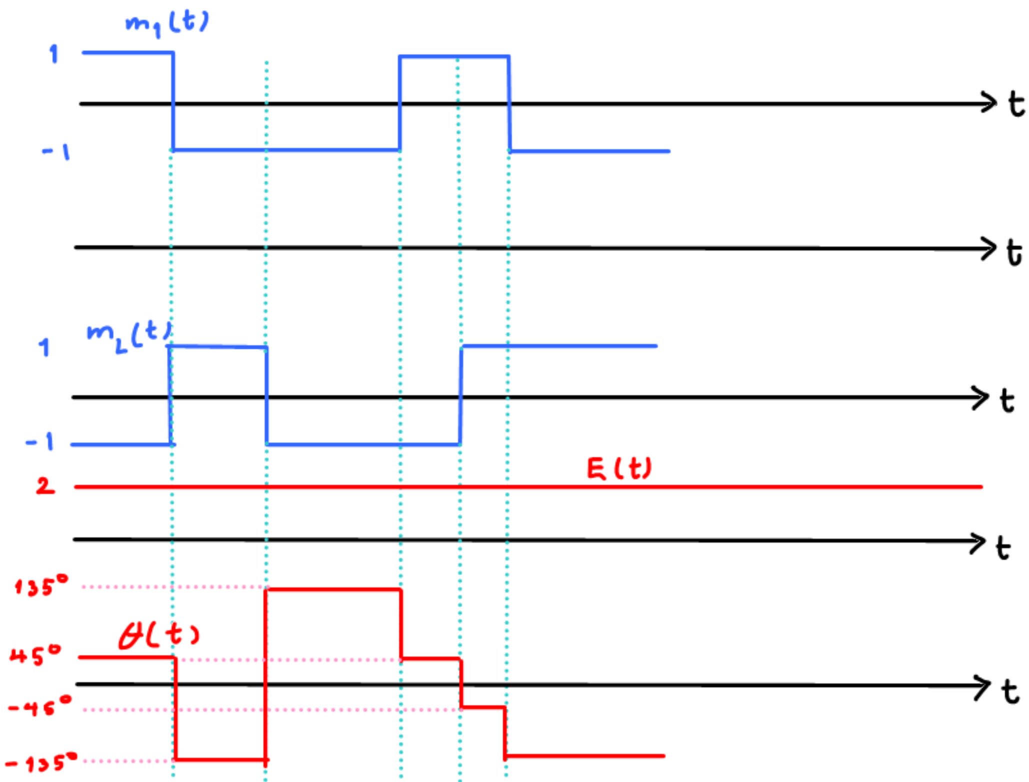
$m_1(t)$ and $m_2(t)$ are plotted below.

We want to express $x_{QAM}(t)$ in the form

$$x_{QAM}(t) = E(t)\cos(2\pi f_c t + \theta(t))$$

where $E(t) \geq 0$ and $\theta(t) \in (-180^\circ, 180^\circ]$.

Plot $E(t)$ and $\theta(t)$.



m_1	m_2	$m_1 - jm_2$	$\sqrt{2}(m_1 - jm_2)$
1	1	$1 - j = \sqrt{2}\angle -45^\circ$	$2\angle -45^\circ$
1	-1	$1 + j = \sqrt{2}\angle 45^\circ$	$2\angle 45^\circ$
-1	1	$-1 - j = \sqrt{2}\angle -135^\circ$	$2\angle -135^\circ$
-1	-1	$-1 + j = \sqrt{2}\angle 135^\circ$	$2\angle 135^\circ$

Quiz 2 Solution ← when $\sqrt{2}$ is factored out in the expression

Suppose $x_{QAM}(t) = m_1(t)\sqrt{2}\cos(2\pi f_c t) + m_2(t)\sqrt{2}\sin(2\pi f_c t)$.

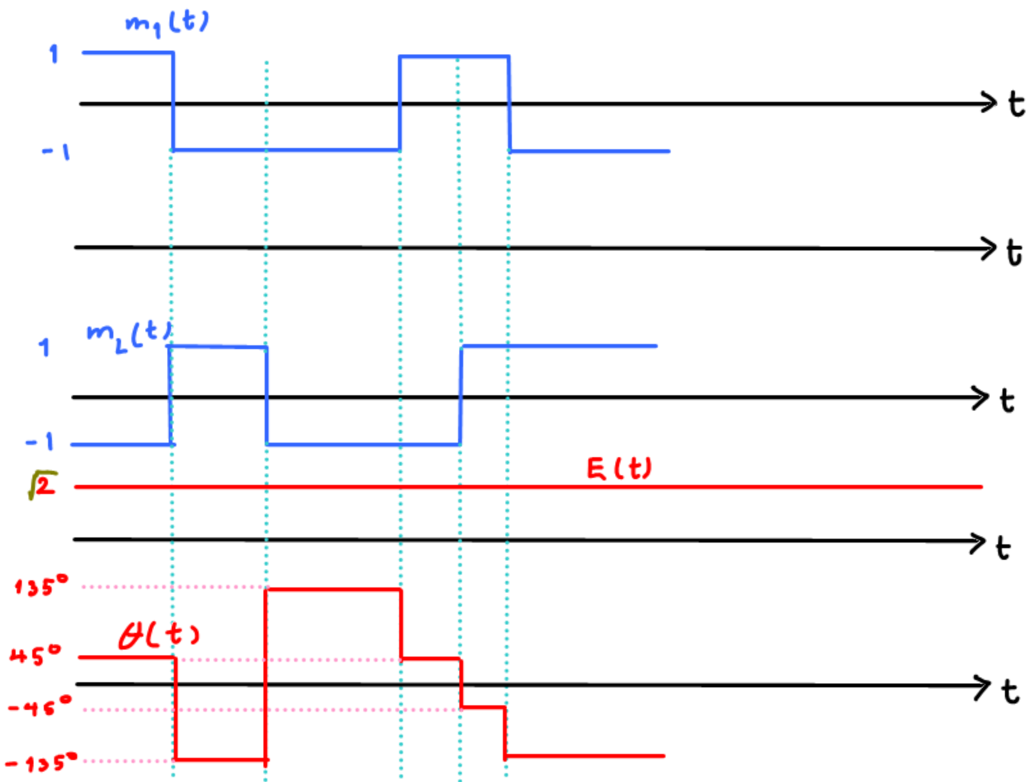
$m_1(t)$ and $m_2(t)$ are plotted below.

We want to express $x_{QAM}(t)$ in the form

$$x_{QAM}(t) = E(t)\sqrt{2}\cos(2\pi f_c t + \theta(t))$$

where $E(t) \geq 0$ and $\theta(t) \in (-180°, 180°]$.

Plot $E(t)$ and $\theta(t)$.



m_1	m_2	$m_1 - j m_2$
1	1	$1 - j = \sqrt{2}\angle -45°$
1	-1	$1 + j = \sqrt{2}\angle 45°$
-1	1	$-1 - j = \sqrt{2}\angle -135°$
-1	-1	$-1 + j = \sqrt{2}\angle 135°$

- $m_1(t)$ and $m_2(t)$ can be separately demodulated.

4.76. Sinusoidal form (envelope-and-phase description [3, p. 165]):

$$x_{\text{QAM}}(t) = \sqrt{2}E(t)\cos(2\pi f_c t + \theta(t)),$$ ← Form #2

where

$$\text{envelope:} \quad E(t) = \sqrt{m_1^2(t) + m_2^2(t)}$$

$$\text{phase:} \quad \theta(t) = -\tan^{-1}\left(\frac{m_2(t)}{m_1(t)}\right)$$

caution; This is trickier than it looks

- The envelope is defined as nonnegative. Negative "amplitudes" can be absorbed in the phase by adding ±180°.

4.77. Complex form:

$\sqrt{2}\,\text{Re}\{(m_1(t) - jm_2(t))(\cos(2\pi f_c t) + j\sin(2\pi f_c t))\}$

$$x_{\text{QAM}}(t) = \sqrt{2}\text{Re}\left\{(m(t))\, e^{j2\pi f_c t}\right\}$$ ← Form #3

where[20] $m(t) = m_1(t) - jm_2(t)$.

$= \sqrt{2}m_1(t)\cos(2\pi f_c t) + \sqrt{2}m_2(t)\sin(2\pi f_c t)$ ← Form #1

- We refer to $m(t)$ as the ***complex envelope*** (or ***complex baseband signal***) and the signals $m_1(t)$ and $m_2(t)$ are known as the ***in-phase*** and ***quadrature(-phase)*** components of $x_{\text{QAM}}(t)$.

- The term "quadrature component" refers to the fact that it is in phase quadrature ($\pi/2$ out of phase) with respect to the in-phase component.

- Key equation:

$$\text{LPF}\left\{\underbrace{\left(\text{Re}\left\{m(t) \times \sqrt{2}e^{j2\pi f_c t}\right\}\right)}_{x(t)} \times \left(\sqrt{2}e^{-j2\pi f_c t}\right)\right\} = m(t).$$

[20] If we use $-\sin(2\pi f_c t)$ instead of $\sin(2\pi f_c t)$ for $m_2(t)$ to modulate,

$$\begin{aligned} x_{\text{QAM}}(t) &= m_1(t)\sqrt{2}\cos(2\pi f_c t) - m_2(t)\sqrt{2}\sin(2\pi f_c t) \\ &= \sqrt{2}\,\text{Re}\left\{m(t)\, e^{j2\pi f_c t}\right\} \end{aligned}$$

where

$$m(t) = m_1(t) + jm_2(t).$$

4.78. Three equivalent ways of saying exactly the same thing:

(a) the complex-valued envelope $m(t)$ complex-modulates the complex carrier $e^{j2\pi f_c t}$,

- So, now you can understand what we mean when we say that a complex-valued signal is transmitted.

(b) the real-valued amplitude $E(t)$ and phase $\theta(t)$ real-modulate the amplitude and phase of the real carrier $\cos(2\pi f_c t)$,

(c) the in-phase signal $m_1(t)$ and quadrature signal $m_2(t)$ real-modulate the real in-phase carrier $\cos(2\pi f_c t)$ and the real quadrature carrier $\sin(2\pi f_c t)$.

4.79. References: [3, p 164–166, 302–303], [13, Sect. 2.9.4], [4, Sect. 4.4], and [8, Sect. 1.4.1]

4.80. Question: In engineering and applied science, measured signals are real. Why should real measurable effects be represented by complex signals?

Answer: One complex signal (or channel) can carry information about two real signals (or two real channels), and the algebra and geometry of analyzing these two real signals as if they were one complex signal brings economies and insights that would not otherwise emerge. [8, p. 3]